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Indirect measurement of surface temperature and heat flux: optimal design using convexity analysis

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Abstract—A method—convexity analysis—is presented for optimizing the design of indirect measurements involving ill-posed inverse heat conduction problems. Convexity analysis yields a concise, quantitative and physically meaningful assessment of any proposed measurement design. One is thus able to make rational design decisions. Typical questions addressed by convexity analysis in inverse heat conduction problems are: What is the best deployment of a given number of measurements? How does the resolution capability deteriorate as the design is altered from the optimum? What is the utility of the marginal measurement—by how much will the resolution improve if an additional measurement is employed? Quantitative answers to these and other design questions are obtained by implementation of an efficient computerizable min-max algorithm. Design optimization by no means replaces the need for interpretational sophistication, but rather ameliorates the analysis of ill-posed inverse problems.

1. INTRODUCTION

INDIRECT measurement of surface temperature or surface heat flux is common in many heat transfer studies. Surface conditions are typically calculated from transient temperature measurements in an interior point of a body in order to avoid a possible influence of the measuring device on the surface properties. Indirect measurements of surface temperature are encountered, for instance, in calorimetry devices, in special flow problems where direct surface temperature measurement is not feasible, and in aerodynamic heating measurements where temperature instrumentation is buried within an external skin.

The problem of interpreting data obtained at an interior point in a solid, in order to determine a transient surface temperature or surface heat flux, belongs to a special class of problems known as inverse heat conduction problems. In a general sense the inverse problems are associated with the determination of the sources of a temperature or heat-flux field from values of the field at certain points [1]. Inverse problems are distinct from direct heat conduction problems. In direct problems, the heat sources are known as well as either the temperature or heat flux on the surface, and a solution is sought for the temperature field inside the body.

In its general form an inverse heat conduction problem is likely to be ill-posed. That is, it may have infinitely many distinct solutions. The prediction of surface conditions from internal temperature measurements is, therefore, mathematically unstable. Nevertheless, several mathematical schemes have been reported [1–8] which exploit an approximation to the surface conditions, in order to arrive at an approximate 'reasonable' prediction of the surface temperature. On the other hand, an alternative and complementary possibility has been largely overlooked. Namely, the possibility of optimizing the design of the measurement system itself, in order to minimize the range of possible solutions and thus reduce the instabilities inherent in the inverse heat conduction calculations. The optimization of design parameters, such as the number and physical location of the measurement points, has not been systematically studied.

The objective of this paper is to present a mathematical approach for optimizing the design of an indirect measurement of temperature and heat flux. Both steady-state and transient systems are studied. It utilizes the technique of convexity analysis developed by one of the authors [9], to provide concise, quantitative and physically meaningful answers to practical measurement–design questions. For example, the tools of convexity analysis will be used to evaluate the capability of a given deployment of N thermocouples to resolve the total power of an unknown distributed heat source, to establish the best deployment of the thermocouples and to determine the utility of the marginal (Nth) measurement. Convexity analysis is a method for optimization of the measurement–design, and does not replace existing techniques for data interpretation. However, by explicitly optimizing the design with respect to the ambiguity inherent in an ill-posed inverse problem, the difficulty and uncertainty associated with data interpretation can be reduced.

Section 2 presents the results of convexity analysis for three simple examples, without discussing how these results are achieved. The aim of Section 2 is to

NOMENCLATURE

a	inverse of the Biot number
$C(u)$	complete response set to arbitrary distribution in X of u watts, equation (9)
$D(u)$	line of distinguishability
f	point source response function
F	point source response set
h	heat transfer coefficient
k	thermal conductivity
L	length
q	linear power density
q''	heat flux
r	unitless radial coordinate
r_c	characteristic heat flux
$r(t)$	variable heat flux
T	temperature
t	time
u	power source intensity, end-point temperature
v	power source intensity, end-point temperature
X	spatial interval
x	axial coordinate
z	relative power resolution of a measuring system.

Greek symbols

α	thermal diffusivity
β	coefficient, equation (29)
$\delta(x)$	Dirac delta function
ζ	unitless axial coordinate
θ	dimensionless temperature
τ	dimensionless time, Fourier number.

show the reader what convexity analysis can achieve, before subjecting him to a discussion of how to do convexity analysis. Section 3 contains a succinct explanation of the methodology of convexity analysis. In Section 4 is an analysis of a transient heat conduction problem.

2. PRELIMINARY EXAMPLES

Inverse heat conduction problems may be encountered in transient as well as in steady-state systems. An ill-posed inverse problem arises when the information regarding the temperature field is incomplete. For instance, the determination of the total steady-state power generated inhomogeneously in a body is considered a well-posed direct heat conduction problem if the temperature field and its spatial derivatives in the body are known. However if, due to practical limitations, the temperature field is known only at a finite number of points in the body, the problem displays the characteristic features of an ill-posed inverse problem. In the following, a few examples will be discussed to demonstrate the concepts of convexity analysis for the optimization of measurement–design for steady-state inverse heat conduction problems.

The resolution capability of a given deployment of N temperature measurements is expressed by the relative power resolution, z. The relative power resolution of a given measuring system is defined as follows: Any spatial distribution of total power u may be distinguished from any spatial power distribution of total power v if and only if

$$v > zu.$$

For example, if $z = 1.2$ for a given deployment of N temperature measurements, then any spatial distribution of u watts can be distinguished, on the basis of the N measurements, from any spatial distribution of v ($>u$) watts if and only if $v > 1.2u$. The technique of convexity analysis enables precise and efficient evaluation of z. Clearly z is always greater than or equal to unity. The closer z is to 1, the better is the power-resolution capability. It is important to understand that the relative power resolution exceeds unity due to the ill-posed nature of the inverse problem, and not due to any statistical uncertainty of the measurements. That is, the relative power resolution exceeds unity because different spatial distributions of the power density, and hence different values of the total power, are consistent with a given finite set of temperature measurements. Let us now consider some simple examples.

Example 1. Linear heat conduction

In this example we consider one-dimensional flow of heat in a solid bounded by a pair of planes at $x = 0$ and L. Let $q(x)$ be a steady-state power density in the interval $X = [0, L]$. We shall suppose that $q(x)$ may assume any arbitrary shape on X. We are going to make N temperature measurements in the interval X, and try to deduce the total power. The boundary conditions are that the heat flux at $x = 0$ and L are known functions of the temperature

$$q''(0) = h[T(0) - T_0] \tag{1}$$

$$q''(L) = -h[T(L) - T_0] \tag{2}$$

where T_0 and h are known constants.

Even though we are allowing $q(x)$ to assume any arbitrary shape, it is necessary only to find the temperature profile in response to a single point source of heat of unit intensity, in order to determine the relative power resolution. It is unnecessary to solve the heat conduction equations for any more complicated power distribution. (The reason for this will become clear in Section 3.) In other words, in order to determine the relative power resolution for a given deployment of N temperature measurements, it is sufficient to find a Green's function for the Laplacian operator. Thus we must solve

$$\frac{d^2T}{dx^2} = -\frac{1}{k}\delta(x-x') \tag{3}$$

where k is the thermal conductivity and $\delta(x)$ is the Dirac δ function. In addition, because the system is at equilibrium, the heat transfer at the ends of the solid must equal the rate of heat production. Consequently we have the additional condition

$$q''(0)+q''(L) = 1.$$

We shall employ the following unitless quantities:

$$\zeta = x/L \quad \text{and} \quad a = k/hL$$

where we recognize that a is the inverse of the Biot number. The solution of equation (3) is

$$\frac{k}{L}[T(\zeta)-T_0] = \begin{cases} \dfrac{a+1-x'/L}{2a+1}(\zeta+a), & 0 \leqslant \zeta \leqslant x'/L \\ \dfrac{a+x'/L}{2a+1}(1+a-\zeta), & x'/L \leqslant \zeta \leqslant 1 \end{cases}. \tag{4}$$

At a given location ζ the temperature can vary according to the source location, x'/L. Considering for instance the first part of equation (4), a minimum temperature is obtained at ζ when $x'/L = 1$ and a maximum temperature is obtained for $x'/L = \zeta$. Similarly, in the second part of equation (4), a maximum temperature is obtained for $x'/L = \zeta$ and a minimum value is obtained for $x'/L = 0$. Since the temperature depends linearly on the source power, the relative power resolution, z, is the ratio between the maximum and minimum temperature, i.e.

$$z(\zeta) = \begin{cases} \dfrac{a+1-\zeta}{a}, & 0 \leqslant \zeta \leqslant 1/2 \\ \dfrac{a+\zeta}{a}, & 1/2 \leqslant \zeta \leqslant 1 \end{cases}. \tag{5}$$

We wish to select the best possible value for the (normalized) position of the measurement, ζ. That is, we wish to select the value of ζ which minimizes the relative power resolution, $z(\zeta)$. In Fig. 1 we show $z(\zeta)$ vs ζ for various values of a. For example, if $a = 1$ and $\zeta = 0.2$, then the relative power resolution is 1.8. This means that a single measurement at position 0.2 is able to distinguish any spatial distribution in X of u watts from any spatial distribution of more than $1.8u$ watts. Furthermore, for any power between u and $1.8u$ watts, there are some spatial distributions at that power whose temperature at point $\zeta = 0.2$ is the same as the temperature due to some spatial distribution of u watts.

From equations (5) we see that the best position for a single measurement (i.e. minimum z) is at $\zeta = 0.5$. In Fig. 2 we show $z(0.5)$ vs k/hL. This shows that as the heat transfer coefficient at the ends of the line source becomes small as compared to the heat conduction in the source, the relative power resolution improves. This is reasonable, since $k \gg hL$ means that the temperature profile along the rod is insensitive to the power distribution.

To summarize the one-measurement resolution capability, we note that equations (5) indicate the best measurement position, assess the resolution at this optimum point, and give a rigorous quantitative evaluation of the reduction in resolution capability as the measurement is removed from the optimum location.

The relative power resolution of a single measurement is not very good unless the Biot number is very small. The method of convexity analysis enables ready evaluation of the relative power resolution for multiple measurements. Consider a symmetric two-measurement arrangement

$$0 \leqslant \zeta_1 \leqslant 1/2 \quad \text{and} \quad \zeta_2 = 1-\zeta_1.$$

In Fig. 3 we show the relative power resolution vs ζ_1 for $k/hL = 1$. We note that when the detectors are positioned at the ends of the source ($\zeta_1 = 0$, $\zeta_2 = 1$), the relative resolution is unity. This is in fact an analytically precise result for any value of k/hL. It means

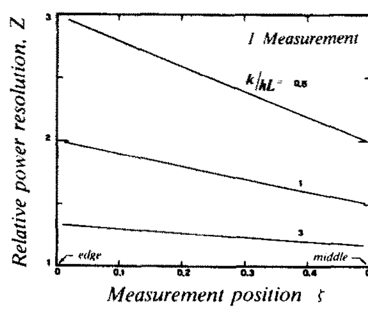

FIG. 1. Relative power resolution vs the position of a single measurement (Example 1).

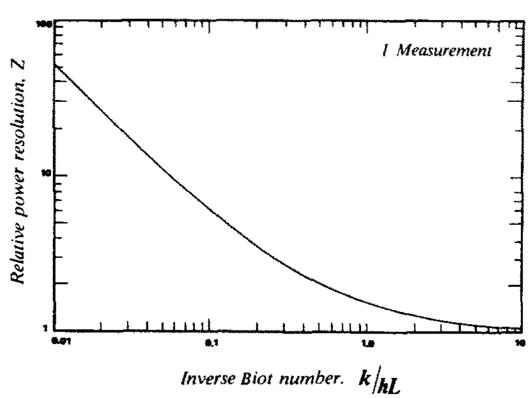

FIG. 2. Relative power resolution at the optimum measurement position vs the inverse of the Biot number (Example 1).

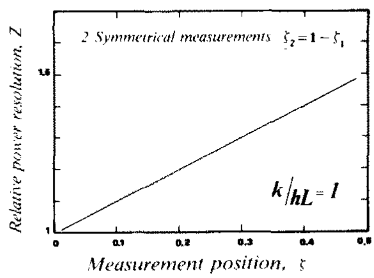

FIG. 3. Relative power resolution vs the measurement position for a pair of symmetrical measurements (Example 1).

that by measuring the end-point temperatures, any spatial distribution of u watts can be distinguished from any spatial distribution of $v > u$ watts. The total power is thus completely resolvable. Furthermore, the results shown in Fig. 3 allow precise quantitative evaluation of the degradation of the resolution capability as the measurements are withdrawn from the end-points. For example, if $\zeta_1 = 0.1$ and $\zeta_2 = 0.9$, then the relative power resolution is 1.1. This indicates that any spatial distribution of u watts can be distinguished from any spatial distribution of v watts if and only if $v > 1.1u$. Let us now proceed to a slightly more complicated situation.

Example 2. Line source in an infinite medium

Consider an arbitrary steady-state distribution of power along the x_1-axis in the interval $X = [-L, L]$ of an infinite medium and let the heat be conducted isotropically in three dimensions. For this formulation the Green's function is particularly easy to find. We note that the temperature field in this example is entirely different from that in example 1, even though both cases deal with one-dimensional distribution of the heat source.

Let us first consider a single measurement at a point in three-dimensional space, $x = (x_1, x_2, x_3)$. Since the heat source is distributed only on the x_1-axis, the problem has cylindrical symmetry. Define

$$\zeta = x_1/L \quad \text{and} \quad r^2 = [x_2^2 + x_3^2]/L^2.$$

The relative power resolution for this single measurement is

$$z(\zeta, r) = \begin{cases} \sqrt{\left(\dfrac{r^2+(\zeta+1)^2}{r^2}\right)}, & 0 \leqslant \zeta \leqslant 1 \\ \sqrt{\left(\dfrac{r^2+(\zeta+1)^2}{r^2+(\zeta-1)^2}\right)}, & 1 \leqslant \zeta \end{cases} \tag{6}$$

The aim of our analysis, as in the previous example, is to choose the measurement position, ζ. In Fig. 4 we show the relative power resolution vs ζ for various radial positions r. Interestingly, z has a local optimum at $\zeta = 0$ (thermocouple located above the centre of the line source). It is significant that, for fixed r, the detector must be moved to a point far beyond the end of the heat source before z regains the value which it has over the centre of the source. For example,

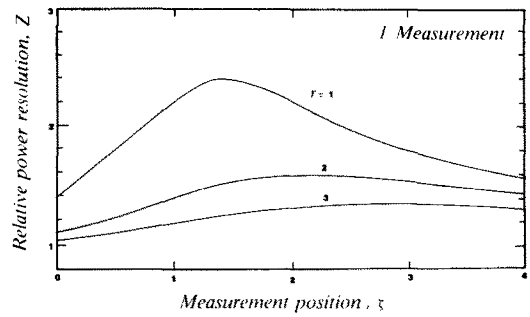

FIG. 4. Relative power resolution vs the measurement position of a single measurement (Example 2).

$z(\zeta = 0, r = 1) = 1.4$, while $z(\zeta = 5, r = 1) = 1.5$. In practice there are likely to be limits on how far the measurement can be removed from the heat source. If the relative power resolution obtainable in practice with a single measurement is unsatisfactory, we should consider two or more measurements. Let us first consider two symmetrical measurements. Thus

$$\zeta_1 = -\zeta_2 \quad \text{and} \quad r_1 = r_2.$$

In Fig. 5 we show the relative power resolution for two symmetrical measurements vs the axial position above the heat source. We note that the use of two measurements enables much closer radial positioning of the measurements. Also we see a surprising local optimum in the relative power resolution at about $\zeta = 0.6$. The value of this local optimum is

$$z[\zeta_1 = -\zeta_2 = 0.63, \quad r_i = 0.2] = 1.95$$

$$z[\zeta_1 = -\zeta_2 = 0.70, \quad r_i = 0.5] = 1.18.$$

In the latter case, e.g. any spatial distribution of u watts is distinguishable from any spatial distribution of $v > u$ watts if and only if $v > 1.18u$.

We can also consider three or more measurements, and we can obtain a quantitative assessment of the improvement resulting from these additional measurements. For two measurements at $r_i = 0.2$, z has a local maximum (poor resolution) near $\zeta_1 = 1$. What happens if we add more measurements? Sample results are

one measurement:
$z(\zeta_1 = 1, r = 0.2) = 10.05$

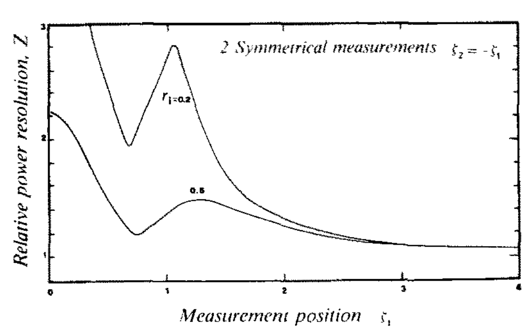

FIG. 5. Relative power resolution vs the measurement position for two symmetrical measurements (Example 2).

two measurements:

$$z(\zeta_1 = -\zeta_2 = 1, r_i = 0.2) = 2.80$$

three measurements:

$$z(\zeta_1 = -\zeta_2 = 1, \zeta_3 = 0, r_i = 0.2) = 1.53$$

four measurements:

$$z(\zeta_1 = -\zeta_2 = 1, \zeta_3 = -\zeta_4 = 0.3, r_i = 0.2) = 1.24.$$

These results are very incomplete, and we should really seek the optimum deployment of two, three and four measurements, rather than just this arbitrary selection of thermocouple locations. However, we see a pattern which is quite common in the analysis of spatially random phenomena: the utility of the marginal detector decreases rapidly as the number of measurements rises. Two measurements are much better than one, and three are considerably better than two, but four measurements do not provide a dramatic improvement on three. Conversely, a large number of measurements is likely to be required to get z really close to unity.

We must stress again that in this example we have made no systematic attempt to find the true optimum deployment of the detectors. However, the technique of convexity analysis enables overall quantitative optimization, and examination of Fig. 5 shows that proper detector deployment can be quite beneficial.

Example 3. Constrained power distributions

In the previous examples we have allowed u watts of power to assume any arbitrary distribution in a specified segment of one coordinate in space. The methodology of convexity analysis is fully capable of handling situations where the heat source is distributed in a disc, or sphere, or in any defined region no matter how irregular. In fact once the Green's function for the Laplacian operator has been found, no particular difficulty arises in considering any given spatial domain for the heat source.

Convexity analysis has an additional dimension of generality. Rather than allowing the power to assume any conceivable distribution in a specified spatial domain X, we may wish to consider constraints on the range of variability of the spatial distribution of the power. For instance, the power density may be uniformly bounded throughout the domain X, or the power density may increase or decrease monotonically in certain specified directions.

For instance, let us consider again example 2, where the domain of the heat source is an interval on the x_1-axis, $X = [-L, L]$, and the heat is conducted isotropically in three dimensions. Let u watts of power be distributed in X according to

$$q(x) = u\frac{\pi}{4L}\cos\frac{\pi x}{2L} + c\frac{\pi}{4L}\sum_{m=1}^{M} b_m \sin\frac{m\pi x}{L} \quad \mathrm{W\,cm^{-1}}$$

where M is a fixed positive integer, c is a positive constant (not necessarily small), and the parameters b_m vary independently on the unit interval

$$0 \leqslant b_m \leqslant 1, \quad m = 1, 2, \ldots, M.$$

Note that the total power of the distribution $q(x)$ is u watts, regardless of the value of the vector b. Consequently, different values of the vector $b = (b_1, \ldots, b_M)$ cause the same total power to generate different temperature fields. Conversely, different power distributions of different total power (different values of u) can generate identical temperatures at a fixed set of measurement positions.

As in the previous examples, one is able to evaluate the relative power resolution for any deployment of N measurements. It is easily shown that a single measurement performed at $\zeta = 0$ and at any value of r yields a relative power resolution of unity. Furthermore, a complete analysis will reveal the degree of degradation of the resolution capability as the detector is moved off centre; the improvement as more measurements are made; and so on.

3. A BRIEF EXPLANATION OF CONVEXITY ANALYSIS

The results which we have presented in the last section and which will be presented in a more detailed example in Section 4, are based on a rigorous mathematical analysis. The numerical calculations employ a simple and efficient min–max algorithm which enables the analysis of an unlimited variety of measurement designs. Very large measurement multiplicity is readily handled, as is a wide range of variability of source-term geometry. In order to give these claims—of mathematical rigour and computational efficiency—some plausibility, we shall present a skeleton outline of the methodology of convexity analysis. A completely general and formal presentation would be confusing for the uninitiated reader, so we shall formulate the presentation in the context of a specific class of problems. (For greater detail see refs. [9, 10]). We shall consider the design of a measurement whose aim is to determine the total power of an unknown spatially distributed heat source, based on a finite number of temperature measurements.

We must begin with some definitions. The spatial domain of the heat source is represented by the set X. That is, any point at which heat may be generated is represented by an element of the set X. Similarly, the spatial domain of the measurements is represented by the set X': any point at which a temperature measurement may be performed is an element of X'. The sets X and X' may or may not overlap. Let us imagine a single heat source of unit intensity concentrated at a point x in X. The resulting temperature at a point x' in X' is given by the *point source response function*, $f(x, x')$. The point source response function is subject to the boundary conditions of the specific problem in question, and is very closely related to the Green's function for the Laplacian operator. If we are considering N temperature measurements at points $x'_1, \ldots, x'_N$, then we let f be a *vector point source response function*, and we usually only denote the position of the point source

$$f(x) = [f(x, x'_1), \ldots, f(x, x'_N)]. \tag{7}$$

Convexity analysis is based on the analysis of sets of vectors. The most fundamental set is the *point source response set*, F. This is the set of all vector values which $f(x)$ may assume. Formally

$$F = \{f(x), \quad \text{for all } x \text{ in } X\}. \tag{8}$$

F is a set of N-vectors: all those collections of N temperature measurements (N-vector responses) for any positioning in X of a single point source of heat of unit power.

The next set encountered in convexity analysis is the *complete response set*, C. This is the set of all N-vector responses to any arbitrary distribution in X of one unit of power. The complete response set is a set of vectors, like the point source response set. C contains all collections of N temperature measurements which may be obtained in response to any arbitrary spatial distribution within X of one unit of power. (Let us henceforth adopt the watt as our unit of power.)

Because of the linear superposition of the responses to different heat sources, the set of all N-vector responses to u watts—the complete response set for u watts, $C(u)$—is obtained by simply multiplying each element of C by the scalar u. Thus $C(u)$, the set of all vector responses to an arbitrary distribution in X of u watts, is

$$C(u) = uC. \tag{9}$$

Before stating the relationship between F and C, we make a short digression to define the concept of convexity. A set is convex if, given any two points in the set, the straight line joining them is entirely in the set. For example, elliptical and rectangular regions (boundary and interior) are convex sets of points in the plane, while crescents, L-shapes and U-shapes are not convex. For any set, A, the *convex hull* of A is the smallest convex set containing A. The convex hull of A is denoted $\mathrm{ch}(A)$. For example the convex hull of a v-shaped curve is the triangular region (boundary and interior) obtained by closing the v at the top.

A very important result, known as the convexity theorem, states that the complete response set C is precisely the convex hull of the point source response set F. This theorem is the cornerstone of convexity analysis.

A necessary and sufficient condition for any spatial distribution in X of u watts to be distinguishable (by measurement at N specified positions) from any spatial distribution of v watts is that the complete response sets, $C(u)$ and $C(v)$, be disjoint

$$uC \cap vC = \varnothing. \tag{10}$$

The optimum design is the one for which u and v watts are distinguishable for the greatest range of values of u and v. We shall now discuss a very important property of the complete response sets, which assists in determining the range of distinguishable values of u and v. Suppose u and v watts are always distinguishable, where $u < v$. Then u and v' watts are also always distinguishable, for all $v' > v$. This results from the convexity of the complete response sets, and is not necessarily true for arbitrary (non-convex) sets. This property leads us to define, as before, the *relative power resolution* as the smallest number, z, such that u and v watts are always distinguishable if and only if

$$v > zu \tag{11}$$

Furthermore, a geometrical study of the disjointness of convex sets leads to the following min–max algorithm for evaluating the relative power resolution, z

$$z = \min_{w \in W} \max_{f, g \in F} \frac{\langle w, f \rangle}{\langle w, g \rangle} \tag{12}$$

where $\langle w, f \rangle$ is the inner product of the N-tuple $w = (w_1, \ldots, w_N)$ with the N-vector point source response function. That is

$$\langle w, f \rangle = \sum_{i=1}^{n} w_i f(x, x'_i). \tag{13}$$

W is the set of all real N-tuples for which $\langle w, f \rangle$ is nonzero and of the same sign for all f in F.

The implementation of this algorithm is by a two-stage iteration. First choose an N-tuple w. For example, start with $w_1 = \ldots = w_N = 1$. Then find the values f' and f'' in F for which $\langle w, f \rangle$ is respectively a minimum and a maximum. If $\langle w, f' \rangle$ and $\langle w, f'' \rangle$ are of the same sign, then the N-tuple w which was chosen actually belongs to W. Now alter the N-tuple w and again search on F for the minimum and maximum values of $\langle w, f \rangle$. This double iteration (on W and on F) is continued until a minimum is obtained for the maximum ratio $\langle w, f'' \rangle / \langle w, f' \rangle$. This 'minimum of the maximum' is precisely the relative power resolution. We should stress that only the point source response function is needed in order to evaluate the relative power resolution for arbitrary spatial distributions of the heat source.

Up to now we have assumed that u watts of thermal power may be distributed in the region X according to *any conceivable* power density function. We have imposed no constraints on the set of possible power density functions (other than the implicit assumption that the power density is integrable in X). In many applications such variability of the distribution of power density is realistic. In such cases, the technique for evaluating the relative power resolution which we have presented is suitable. In other applications, however, the spatial distribution of the power density may be constrained in some way. For example, there may be an upper bound on the value which the local power density may attain. One would nevertheless like to evaluate the relative power resolution for the specific set of realizable power density functions. The method by which convexity analysis achieves this is explained through an example in Section 4.

4. CONVEXITY ANALYSIS OF A TRANSIENT INVERSE PROBLEM

4.1. *Formulation*

In a transient measurement one is typically interested in calculating the transient surface temperature based on temperature measurements at one or more points in the body. We shall consider a one-dimensional case consisting of a rod of length $2L$ which is initially at temperature zero. At time $t > 0$ a varying heat flux, $r(t)$, is applied at its ends. It is required to determine the temperature at the ends of the rod based on temperature measurements in the rod. For constant thermal properties, the heat conduction equation and boundary conditions are

$$\frac{\partial T}{\partial t} = \alpha^2 \frac{\partial^2 T}{\partial^2 x} \tag{14}$$

$$T(x,0) = 0 \tag{15}$$

$$-k\left(\frac{\partial T}{\partial x}\right)_{x=\pm L} = r(t) \tag{16}$$

where x is the position on the rod, $|x| \leqslant L$, t the time, $T(x,t)$ the temperature at x, t, and $\alpha^2 = k/\rho C_p$, the thermal diffusivity.

An inverse problem arises when the temperature $T(x,t)$ at one or more points in the body is specified rather than the heat flux $r(t)$. If $r(t)$ were completely known, then equations (14)–(16) could be solved and the temperature at $x = L$ would be known. However, if $r(t)$ is not known, then a finite set of measurements in $|x| < L$ do not necessarily determine the temperature at the end of the rod, $x = L$. Consequently, determination of the temperature at $x = L$ by measuring the temperature in $|x| < L$, is plagued by an unavoidable uncertainty. The aim of this section is to show how convexity analysis provides a quantitative means for evaluating this uncertainty, as a function of the number and locations of the measurement points. This evaluation of the uncertainty serves as a basis for rational decisions and system optimization. It will be shown, for instance, that two measurements provide much better resolution of the temperature at the edge than do either of the single measurements alone. To arrive at a quantitative measure of the ability to determine the edge temperature (at $x = L$) we will first solve the direct problem. Using Duhamel's theorem [11] it can be shown that equations (14)–(16) are solved by

$$\theta(\zeta,\tau) = 2\sum_{n=1}^{\infty}(-1)^n \sin\frac{2n-1}{2}\pi\zeta \times \int_0^{\tau} r(s)\exp\left[-\left(\frac{2n-1}{2}\pi\right)^2(\tau-s)\right]\mathrm{d}s \tag{17}$$

where τ is the dimensionless time or Fourier number, $t(\alpha/L)^2$, θ the dimensionless temperature, $Tk/(Lr_c)$, ζ the dimensionless length, x/L, and r_c a characteristic heat flux at the edge. An important property displayed by equation (17) is that the temperature, θ, is linear and homogeneous in the heat flux r.

Any given heat flux function, $r(s)$ for $0 \leqslant s \leqslant t$, generates a temperature profile at time t according to equation (17). The heat flux function $r(s)$ is unknown. In fact, r may be any function from among a set of functions. Let R represent the set of all physically allowable r functions. Then, different r functions in R will generate different temperature profiles. For any function r in R, let $\theta(\zeta,\tau;r)$ be the corresponding solution. Further, define τ_f to be the dimensionless time for which we wish to estimate the temperature at $\zeta = 1$, and let $R(u)$ represent the largest subset of R for which $\theta(1,\tau_f;r) = u$. That is

$$R(u) = \{r : \theta(1,\tau_f;r) = u, \quad \text{for all } r \in R\}. \tag{18}$$

In many situations we will be able to represent $R(u)$ as the convex hull of a set $G(u)$ of simpler functions

$$R(u) = \mathrm{ch}\,[G(u)]. \tag{19}$$

We shall perform temperature measurements at M space–time points which are denoted

$$p = (p_1, p_2, \ldots, p_M) \quad \text{where} \quad p_m = (x_m, t_m).$$

The M-vector measurement in response to a given function r is

$$\theta(r) = [\theta(p_1;r), \ldots, \theta(p_M;r)].$$

The complete response set for $\theta(1,t_f) = u$ is the set of all measurements obtainable at the M space–time points p, for which the end-point temperature is u. Specifically, this complete response set is

$$\Theta(u) = \{\theta(r), \text{for all } r \in R(u)\}. \tag{20}$$

Let us define the fundamental response set for end-point temperature $\theta(1,t_f;r) = u$ as

$$F(u) = \{\theta(r), \text{for all } r \in G(u)\}. \tag{21}$$

Because $R(u)$ is the convex hull of $G(u)$, and because $\theta(r)$ is linear and homogeneous in r (see equation (17)), we conclude that

$$\Theta(u) = \mathrm{ch}\,[F(u)]. \tag{22}$$

This relationship will enable a great simplification in the evaluation of the resolution capability.

4.2. *Relative temperature resolution*

The M measurements at space–time positions p are able to distinguish an end-point temperature of u from an end-point temperature of v if and only if the corresponding complete response sets are disjoint

$$\Theta(u) \cap \Theta(v) = \varnothing. \tag{23}$$

Because the complete response sets are convex (and compact—a necessary technicality), a necessary and sufficient condition for this disjointness is that there be a hyperplane which separates the two response sets in the M-dimensional response space. A hyperplane in the response space is the set of all M-vectors θ such that

$$\sum_{i=1}^{m} w_i \theta_i = \text{constant} \quad (24)$$

where the w_i are constant real numbers. Let us adopt the notation that $\langle w, \theta \rangle$ represents the sum of the left-hand side of equation (24).

From these considerations we see that equation (23) is true (and the end-point temperatures u and v are distinguishable) if and only if there is a real M-tuple $w = (w_1, \ldots, w_M)$ such that

$$\max_{\theta \in \Theta(u)} \langle w, \theta \rangle < \min_{\theta \in \Theta(v)} \langle w, \theta \rangle. \quad (25)$$

Relation (25) provides the basis for evaluating the relative temperature resolution for a given set of M space–time measurement positions. The numerical result of such an analysis is a 'line of distinguishability', $D(u)$. An end-point temperature v is distinguishable from the end-point temperature u (for $v > u$) if and only if

$$v > D(u). \quad (26)$$

Because $\Theta(u)$ is the convex hull of $F(u)$, the extrema in relation (25) can be evaluated on $F(u)$ rather than on $\Theta(u)$. This is a basic property of convex sets, and is of very great importance. Furthermore, a search on $F(u)$ for an extremum can be expressed as a search on $G(u)$. These conclusions may be represented as follows:

$$\underset{\theta \in \Theta(u)}{\text{extremum}} \langle w, \theta \rangle = \underset{\theta \in F(u)}{\text{extremum}} \langle w, \theta \rangle \quad (27a)$$

$$= \underset{g \in G(u)}{\text{extremum}} \langle w, \theta(g) \rangle. \quad (27b)$$

It is instructive to compare relations (11) and (26). In Section 3 we considered arbitrary spatial distributions (of the power density), while in the present section we have imposed constraints on the range of spatial variability (of the r-functions). In Section 3 we were able to express the resolution capability with a single number, the relative power resolution z, as in relation (11), while in the presence of constraints on spatial variability we find that the resolution capability is expressed by a curve: the line of distinguishability. Equivalently, we can say that constraints on the range of spatial variability have caused the relative resolution to be a function of the measured parameter, u. Thus we could have written, instead of inequality (26), the following condition for distinguishability:

$$v > z(u)u \quad (28)$$

where $z(u) = D(u)/u$.

4.3. *Convex modelling*

In Section 4.1 we introduced the set R of all allowed r-functions which may occur in equation (16). The physical definition of the problem will suggest the general range of functions which make up the set R. Particular care and caution must be exercised in the precise choice of R. Two considerations are of pre-eminent importance in selecting R. On the one hand, the r-functions introduced must be physically reasonable and R must not exclude any class of physically reasonable functions. On the other hand, it is very desirable that the subsets $R(u)$ be the convex hulls of fundamental sets $G(u)$ of 'simpler' functions. (Recall that $R(u)$ is the greatest subset of R for which the end-point temperature equals u.) The elements of $G(u)$ are sufficiently simple if equation (17) can be evaluated for an arbitrary element of $G(u)$ without too much difficulty. Since $R(u)$ is the convex hull of $G(u)$, it will not be necessary to evaluate θ for any elements of R other than elements of $G(u)$. (See equations (27).) The task of selecting the set R so that it is a sufficiently realistic model of the physical problem, and so that the subsets $R(u)$ are convex, is referred to in general as 'convex modelling'.

Let us consider a specific case. Suppose that the physical definition of the problem suggests that $R(u)$ is a convex set of functions $r(t)$ which increase monotonically in time on the interval $[0, \tau_f]$ and which are bounded by $r' \leqslant r(t) \leqslant r''$. Can we model this problem convexly? That is, can we find sets $G(u)$ of simple fundamental functions such that $R(u) = \text{ch}\,[G(u)]$ and such that $R(u)$ is physically reasonable?

To show how this may be done, consider the fundamental functions

$$g(t; s, \beta) = r' + (r'' - r')\beta U(t-s) \quad (29)$$

where $U(t)$ is the unit step function: equal to 0 for $t < 0$, equal to 1 otherwise. Thus $g(t; s, \beta)$ is a step function which jumps from r' to $r' + (r'' - r')\beta$ at $t = s$. We will always assume that $0 \leqslant \beta \leqslant 1$ and that $0 \leqslant s \leqslant \tau_f$. It is readily shown that all bounded, monotonically increasing functions can be expressed as averages of such step functions.

Recall that an element r in R results in an end-point temperature $\theta(1, \tau_f; r)$. Let $G(u)$ contain those step functions which produce an end-point temperature of u at τ_f. That is, $G(u)$ is

$$G(u) = \{ g(y; s, \beta) : \theta(1, t_f; g) = u \}. \quad (30)$$

Since $\theta(1, \tau_f; r)$ is linear and homogeneous in r, it results that any convex combination of elements of $G(u)$ also produces an end-point temperature equal to u. Thus $\text{ch}\,[G(u)]$ is a rich collection of bounded monotonically increasing functions which result in u as an end-point temperature. It is probably reasonable to adopt $\text{ch}\,[G(u)]$ as the set of all allowed heat transfer functions whose end-point temperature is u at τ_f. That is:

$$R(u) = \text{ch}\,[G(u)]. \quad (31)$$

It remains only to characterize the sets $G(u)$. From equation (17) one finds that $g(t; s, \beta)$ belongs to $G(u)$ if s and β satisfy a certain transcendental equation. It is not too difficult to evaluate the range of values (s, β) for which $g(t; s, \beta)$ belongs to $G(u)$. The problem of convexly modelling the set R is now solved in a reasonably satisfactory manner.

4.4. *Non-linear boundary conditions*

Before continuing with the analysis of this example, let us consider an important consequence of the concept of convex modelling. Cursory examination of equation (16) may leave the mistaken impression that the design optimization procedure can be applied only when the boundary conditions are not functions of the temperature. To demonstrate that this is not so, let us suppose that equation (16) is replaced by

$$-k\left(\frac{\partial T}{\partial x}\right)_{x=L} = r[t, T(L, t)]. \quad (16a)$$

Since T itself is a function of time, the right-hand side of equation (16a) is no more than a function of time. More precisely, $r(t, T)$ may represent a family of functions of time. The fact that we write $r(t, T)$, rather than $r(t)$, is just an expression of our limited knowledge about the form which any particular heat flux function may assume. In those situations for which the design optimization is difficult, this limitation of our knowledge arises from the very great complexity of the phenomenon under examination. However, in optimizing the design of the measurement, we need not be concerned with the form of arbitrary allowable heat flux functions. Rather, we need only know the range of variability of the heat flux functions. $R(u)$ represents the set of heat flux functions which, to the best of our understanding of the problem at hand, may occur. If the phenomenon being studied is complex, then the set $R(u)$ is defined by making some generalizations about the phenomenon. That is, $R(u)$ is to be defined by specifying the properties of its members, rather than by specifying its members explicitly. For example, we may be able to satisfy ourselves that it would not be inconsistent with our understanding to assume that the set $R(u)$ contains certain monotonically increasing functions of time, as in the previous subsection. In order to optimize the design of the measurement, we must be able to specify the sets $R(u)$. Hopefully, we will be able to model them convexly. However, we have no need of knowing the temperature field in response to any particular elements of $R(u)$ other than those elements of the fundamental set whose convex hull equals $R(u)$. If we are able to specify the set $R(u)$ as a set of functions of time only, then our optimization of the design becomes independent of the non-linearity implied in equation (16a).

4.5. *Results*

Results for a specific case of monotonically increasing $r(t)$ in the range from 0 to 500 W m^{-2} are presented in Figs. 6–9. Figure 6 shows the fractional temperature resolution for a single thermocouple located at relative distances of 2, 5 and 10% from the end of the rod, as a function of the dimensionless temperature, θ, at the edge. The maximum heat flux (500 W m^{-2}) was chosen as a characteristic heat flux, r_c, in the definition of the dimensionless temperature. The resolution capability improves as the detection point comes closer to the edge (x/L increases). Obviously the resolution capability is perfect (that is, $D(u) = u$) when the measurement point coincides with the edge of the rod. For a thermocouple at $x = 0.95$ and dimensionless temperature of $\theta = 20$, the fractional resolution is 0.2. This means, for instance, that for an aluminium rod 10 cm long, a single thermocouple located at a distance of 5 mm from the edge can differentiate between end-point temperatures of 5 and 6°C at the edge. The resolution capability improves as the edge temperature increases. For instance using the above example of an aluminium rod, the same thermocouple may be used to differentiate between 10 and 10.8°C. That is, the fractional resolution is 0.08 at a dimensionless temperature of $\theta = 40$ for $x/L = 0.95$.

Figure 7 shows the fractional resolution of the edge temperature at time τ_f, for a single measurement at position $x = 0.95$ and at time τ_f. Two values of the dimensionless temperatures at the edge were used: 20 and 40. The resolution capability deteriorates as the measurement time increases. For a larger τ_f there exists a broader range of r functions in $R(u)$. This increases the range of possible temperatures at the

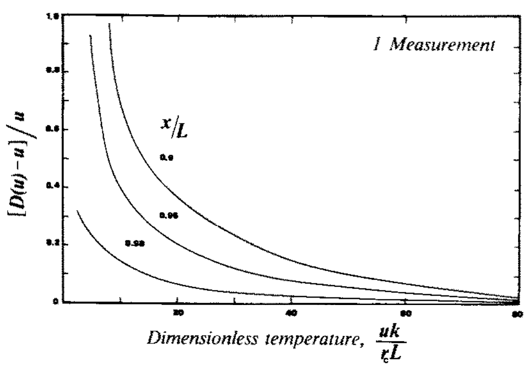

FIG. 6. Fractional temperature resolution, $[D(u)-u]/u$, for a single thermocouple as a function of temperature.

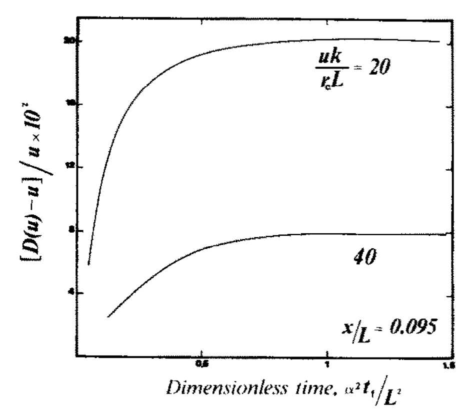

FIG. 7. Fractional temperature resolution, $[D(u)-u]/u$, for a single thermocouple as a function of time.

thermocouple location and thus decreases the detector's resolution capability.

The improvement in the resolution capability obtained by adding a second thermocouple is demonstrated in Figs. 8 and 9 for a dimensionless temperature of 40. Figure 8 shows the fractional temperature resolution for two cases. In the upper curve, one of the two thermocouples is located at $x_1 = 0.95$, and in the lower curve one thermocouple is at $x_1 = 0.98$. The position of the second thermocouple is shown on the abscissa. Adding another thermocouple at x_2 improves the resolution capability dramatically. Considering for instance the first curve of $x_1 = 0.95$, then adding a second measurement at $x_2 = 0.9$ improves the fractional resolution from 0.08 to 0.0013. This means that by using two thermocouples at 0.95 and 0.9 one can differentiate between a temperature of 10 and 10.013°C at the edge of the aluminium rod discussed above. (Recall that one measurement at $x/L = 0.95$ is able to distinguish no better than between 10 and 10.8°C.) The resolution improves if one of the thermocouples is placed closer to the edge. For instance, adding a second thermocouple to a thermocouple at $x_1 = 0.98$ improves the fractional resolution from 0.02 to 0.0006. It is interesting to note that the exact location of the second thermocouple has only a small effect on the relative resolution of the system.

The fractional resolution of two thermocouples at $x = 0.95$ and 0.9 is depicted in Fig. 9 as a function of the dimensionless measurement time. As discussed for a single thermocouple system, the resolution capability deteriorates asymptotically with the dimensionless time.

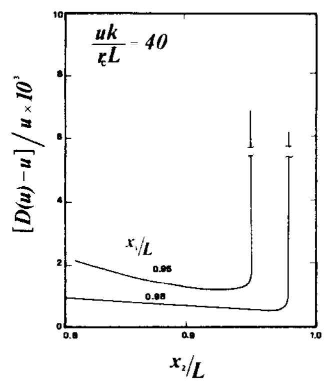

FIG. 8. Fractional temperature resolution vs measurement position for two thermocouples.

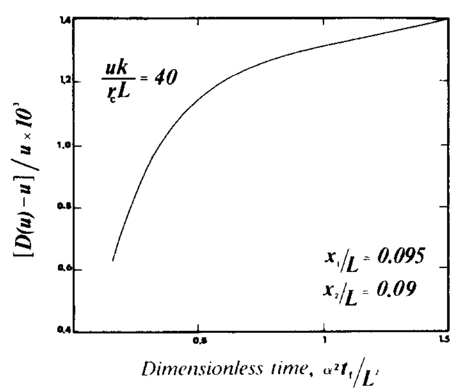

FIG. 9. Fractional temperature resolution for two thermocouples as a function of time.

5. SUMMARY AND CONCLUSIONS

Inverse heat conduction problems typically involve the inference of surface temperatures or surface heat fluxes from measurements at interior points. More generally, inverse problems involve the determination of heat or heat flux source terms, from a finite set of measurements. Such problems can be 'ill-posed': having an infinite number of solutions. That is, the spatial distribution of the source is not necessarily uniquely determined by a finite set of measurements. In fact, the set of possible spatial distributions of the source may be infinite. This spatial uncertainty is a fundamental feature of an ill-posed inverse heat conduction problem.

The spatial uncertainty generates an ambiguity in the interpretation of an ill-posed inverse heat conduction problem. While various data analysis techniques have been developed to reduce the ambiguity in the interpretation of ill-posed problems, our aim has been to present a different though complementary approach. We have discussed a method—convexity analysis—for optimizing the design of the measurement with respect to the spatial uncertainty which lies at the root of the interpretational ambiguity. Design optimization by no means replaces the need for interpretational sophistication, but rather ameliorates the analysis of ill-posed inverse problems.

In this paper we have concentrated solely on the spatial uncertainty inherent in ill-posed inverse heat conduction problems. We recognize that another source of uncertainty—random processes effecting the measurements themselves—may also be important. Convexity analysis is capable of incorporating this 'statistical uncertainty' in the overall design optimization; many design optimization problems (outside the field of heat conduction) involving both spatial and statistical uncertainty have been solved.

Convexity analysis is a very general mathematical tool for design optimization in a broad range of measurement applications. Exploitation of concepts from the theory of convex sets allows the rigorous treatment of the possibly infinite set of spatial distributions of heat or heat flux source terms. Convexity analysis yields a concise, quantitative and physically meaningful assessment of any proposed measurement design. One is thus able to make rational design decisions. Typical questions addressed by convexity analysis in inverse heat conduction problems are: What is the best deployment of a given number of measurements? How does the resolution capability deteriorate as the design is altered from the optimum? What is the utility of the marginal measurement—by how much will the resolution improve if an additional measurement is employed? Quantitative answers to these and other design questions are obtained by implementation of an efficient computerizable min-max algorithm.

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MESURE INDIRECTE DE TEMPERATURE DE SURFACE ET DE FLUX THERMIQUE: OPTIMISATION BASEE SUR L'ANALYSE DE LA CONVEXITE

Résumé—Une méthode, analyse de la convexité, est présentée pour optimiser les mesures indirectes basées sur le problème inverse de la conduction thermique. L'analyse de convexité fournit une évaluation concise, quantitative d'une mesure quelconque. On est ainsi capable de concevoir un montage rationnel. Les questions typiques posées par l'analyse de convexité dans les problèmes inverses de conduction sont: Quel est le meilleur déploiement d'un nombre donné de mesures? Comment se détériore la résolution quand on s'écarte de l'optimum? Quelle est l'utilité de la mesure marginale et de combien s'améliore la résolution si on ajoute une mesure? Des réponses quantitatives à cela et à d'autres questions sont obtenues par un algorithme min–max informatisé. L'optimisation du montage ne supprime pas la sophistication de l'interprétation mais elle améliore l'analyse des problèmes inverses mal posés.

INDIREKTE MESSUNG VON OBERFLÄCHENTEMPERATUR UND WÄRMESTROMDICHTE: OPTIMALE ANORDNUNG UNTER ANWENDUNG DER METHODE DER KONVEXITÄTSANALYSE

Zusammenfassung—Eine Methode—die Konvexitätsanalyse—wird vorgestellt zur Optimierung der Anordnung bei indirekten Messungen, die auch ungünstige inverse Wärmeleitprobleme beinhalten. Die Konvexitätsanalyse liefert eine kurze, quantitativ und physikalisch sinnvolle Einschätzung von beliebigen Meßanordnungen. Es ist damit möglich, begründete Entscheidungen über eine Anordnung zu treffen. Typische Fragen, die durch die Konvexitätsanalyse bei inversen Wärmeleitproblemen beantwortet werden können, sind: Was ist die beste Vorgehensweise bei einer gegebenen Anzahl von Messungen? Wie verschlechtert sich das Auflösungsvermögen, wenn die optimale Anordnung verlassen wird? Was nützt eine Messung, die gerade eben noch möglich ist, um wieviel wird sich die Auflösung verbessern, falls eine zusätzliche Messung durchgeführt wird? Quantitative Antworten auf diese und andere Gestaltungsfragen erhält man aus der Anwendung eines wirkungsvollen, numerisch anwendbaren Minimum–Maximum Algorithmus. Anordnungsoptimierung ersetzt keinesfalls die Notwendigkeit der Interpretation, aber sie verbessert freilich die Analyse von ungünstigen inversen Problemen.

КОСВЕННЫЕ ИЗМЕРЕНИЯ ТЕМПЕРАТУРЫ ПОВЕРХНОСТИ И ТЕПЛОВОГО ПОТОКА: ОПТИМАЛЬНЫЙ РАСЧЕТ ПО МЕТОДУ ВЫПУКЛОГО АНАЛИЗА

Аннотация—Рассматривается метод выпуклого анализа для оптимизации косвенных измерений, в том числе при решении некорректно поставленных обратных задач теплопроводности. Выпуклый анализ позволяет при сокращении количества измерений оценить физическую сущность для любой предложенной техники измерений. Это дает возможность получать рациональные конструкторские решения. Выпуклому анализу адресуются следующие типичные вопросы в обратных задачах теплопроводности: каково оптимально приемлемое число измерений, насколько ухудшается разрешающая способность при отклонении расчета от оптимального, какая целесообразность в расширении предела измерений—насколько улучшится разрешение благодаря дополнительным измерениям? Количественные ответы на эти и другие вопросы получены на компьютере с помощью mini–max алгоритма. Оптимизация расчета ни в коей мере не заменяет точной интерпретации, но сильно улучшает анализ некорректно поставленных обратных задач.